

# Limit shapes for inhomogeneous corner growth models with exponential and geometric weights

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## Abstract

We generalize the exactly solvable corner growth models by choosing the rate of the exponential distribution  $a_i + b_j$  and the parameter of the geometric distribution  $a_i b_j$  at site  $(i, j)$ , where  $(a_i)_{i \geq 1}$  and  $(b_j)_{j \geq 1}$  are jointly ergodic random sequences. We identify the shape function in terms of a simple variational problem, which can be solved explicitly in some special cases.

## 1 Introduction

The corner growth model is a frequently studied model of random growth. See [29] for a detailed introduction, and [22], [28] for an overview of related research. The model describes a cluster of sites that emerges from the corner of a quadrant and grows over time. We represent the quadrant with  $\mathbb{N}^2$  and the cluster with a family of subsets  $S_t \subset \mathbb{N}^2$  indexed by time  $t \geq 0$ . Each site  $(i, j) \in \mathbb{N}^2$  waits for  $(i-1, j)$  if  $i > 1$  and  $(i, j-1)$  if  $j > 1$  to be in the cluster and, after an additional waiting time of  $W(i, j)$ , joins the cluster permanently at time  $G(i, j)$ . More precisely,  $S_t = \{(i, j) \in \mathbb{N}^2 : G(i, j) \leq t\}$  for  $t \geq 0$  and

$$G(i, j) = G(i-1, j) \vee G(i, j-1) + W(i, j) \quad \text{for } i, j \in \mathbb{N}, \quad (1.1)$$

where  $G(i, 0) = G(0, j) = 0$  for  $i, j \in \mathbb{N}$ . As first observed in [23], the preceding recursion implies

$$G(m, n) = \max_{\pi \in \Pi_{1,1,m,n}} \sum_{(i,j) \in \pi} W(i, j) \quad \text{for } m, n \in \mathbb{N}, \quad (1.2)$$

where  $\Pi_{u,v,u',v'}$  is the set of all directed paths from  $(u, v)$  to  $(u', v')$ , that is, all finite sequences  $\pi = ((i_k, j_k))_{1 \leq k \leq l}$  in  $\mathbb{Z}^2$  such that  $(i_{k+1} - i_k, j_{k+1} - j_k) \in \{(1, 0), (0, 1)\}$  for  $1 \leq k < l$ ,  $(i_1, j_1) = (u, v)$  and  $(i_l, j_l) = (u', v')$ . We will refer to the random quantities  $\{W(i, j) : i, j \in \mathbb{N}\}$  and  $\{G(i, j) : i, j \in \mathbb{N}\}$  as *weights* and *last-passage times*, respectively, because of connection (1.2) with directed last-passage percolation. The problem is typically to understand the statistical properties of the last-passage times given the joint probability distribution  $P$  of the weights.

Consider the corner growth model in which the weights are i.i.d. and exponentially distributed with rate  $\lambda > 0$  i.e.  $P(W(i, j) \geq x) = e^{-\lambda x}$  for  $i, j \in \mathbb{N}$  and  $x \geq 0$ . We will refer to this special case as the *exponential model*. In a seminal paper [25], H. Rost observed the equivalence of the exponential model with the totally asymmetric simple exclusion process (TASEP) and proved that

$$\lim_{n \rightarrow \infty} \frac{G(\lfloor ns \rfloor, \lfloor nt \rfloor)}{n} = \frac{(\sqrt{s} + \sqrt{t})^2}{\lambda} \quad \text{for } s, t > 0 \quad P\text{-a.s.} \quad (1.3)$$

He also interpreted (1.3) as an asymptotic shape result in the sense that the rescaled cluster  $S_n/n$  converges as  $n \rightarrow \infty$  to the parabolic region  $\{s, t \in \mathbb{R}_+ : \sqrt{s} + \sqrt{t} \leq \sqrt{\lambda}\}$  in the Hausdorff metric.

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The discrete counterpart of the exponential model is the *geometric model* in which the weights are i.i.d. and geometrically distributed with (fail) parameter  $q \in (0, 1)$  i.e.  $P(W(i, j) \geq k) = q^k$  for  $i, j \in \mathbb{N}$  and  $k \in \mathbb{Z}_+$ . For the geometric model, one can also compute that

$$\lim_{n \rightarrow \infty} \frac{G(\lfloor ns \rfloor, \lfloor nt \rfloor)}{n} = \frac{q}{1-q}(s+t) + \frac{2\sqrt{q}}{1-q}\sqrt{st} \quad \text{for } s, t > 0 \quad P\text{-a.s.}, \quad (1.4)$$

[5], [16], [27]. The existence of the deterministic (nonrandom) a.s. limits in (1.3) and (1.4) follows from standard subadditive arguments, which hold in greater generality, for example, when  $P$  is an arbitrary i.i.d. measure. Define the *shape function*  $g$  by  $g(s, t) = \lim_{n \rightarrow \infty} n^{-1}G(\lfloor ns \rfloor, \lfloor nt \rfloor)$  for  $s, t > 0$   $P$ -a.s. The basic properties of  $g$  such as concavity and homogeneity are also easily derived [29, Theorem 2.1]. An interesting matter is the explicit identification of  $g$ . Despite much effort, this has not been possible except for the exponential and geometric models, which are called *exactly solvable* cases. In this paper, we introduce certain corner growth models with non i.i.d  $P$  whose shape functions exist and can be determined explicitly.

We study *inhomogeneous* generalizations of the exponential and geometric models in which the parameters  $\lambda$  and  $q$  are site-dependent and drawn randomly from an ergodic distribution. More specifically, let  $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$  and  $\mathbf{b} = (b_n)_{n \in \mathbb{N}}$  be stationary random sequences in  $(0, \infty)$  such that the distribution  $\mu$  of  $(\mathbf{a}, \mathbf{b})$  is separately ergodic under the map  $\tau_k \times \tau_l$  for each  $k, l \in \mathbb{N}$ , where  $\tau_k$  is the shift  $(c_n)_{n \in \mathbb{N}} \mapsto (c_{n+k})_{n \in \mathbb{N}}$  for  $k \in \mathbb{Z}_+$ . In particular,  $\mathbf{a}$  and  $\mathbf{b}$  can be independent i.i.d. sequences. Suppose that, given  $(\mathbf{a}, \mathbf{b})$ , the weights are conditionally independent and  $W(i, j)$  is exponentially distributed with rate  $\lambda = a_i + b_j$  for  $i, j \in \mathbb{N}$ . The inhomogeneous geometric model is defined similarly except now  $\mathbf{a}$  and  $\mathbf{b}$  are sequences in  $(0, 1)$  and, given  $(\mathbf{a}, \mathbf{b})$ ,  $W(i, j)$  is geometrically distributed with parameter  $q = a_i b_j$  for  $i, j \in \mathbb{N}$ . Let us write  $\mathbb{P}$  for the joint distribution of the weights and  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  for the joint conditional distribution of the weights given  $(\mathbf{a}, \mathbf{b})$ . To be clear,  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  is a product measure on  $\mathbb{R}_+^{\mathbb{N}^2}$  and  $\mathbb{P}(B) = \int \mathbf{P}_{\mathbf{a}, \mathbf{b}}(B) \mu(d\mathbf{a}, d\mathbf{b})$  for any Borel set  $B \subset \mathbb{R}_+^{\mathbb{N}^2}$ . Identifying  $W(i, j)$  with the projection  $\mathbb{R}_+^{\mathbb{N}^2} \rightarrow \mathbb{R}_+$  onto coordinate  $(i, j)$ ,

$$\mathbf{P}_{\mathbf{a}, \mathbf{b}}(W(i, j) \geq x) = e^{-(a_i + b_j)x} \quad \text{for } i, j \in \mathbb{N} \text{ and } x \geq 0$$

for the exponential model and

$$\mathbf{P}_{\mathbf{a}, \mathbf{b}}(W(i, j) \geq k) = a_i^k b_j^k \quad \text{for } i, j \in \mathbb{N} \text{ and } k \in \mathbb{Z}_+$$

for the geometric model. A noteworthy feature that distinguishes these models from the classical homogeneous counterparts is the correlations of the weights along the rows and the columns, that is,  $W(i, j)$  and  $W(i', j')$  are not independent under  $\mathbb{P}$  if  $i = i'$  or  $j = j'$ .

Our main result is a simple variational description of the shape function. To state it, let  $\alpha$  and  $\beta$  denote the distributions of  $a = a_1$  and  $b = b_1$ , respectively. For the exponential model,

$$g(s, t) = \inf_{z \in [-\underline{\alpha}, \underline{\beta}]} \left\{ s \int_0^\infty \frac{\alpha(dx)}{x+z} + t \int_0^\infty \frac{\beta(dx)}{x-z} \right\} \quad \text{for } s, t > 0, \quad (1.5)$$

where  $\underline{\eta}$  is the left endpoint of the support (the complement of the largest open  $\eta$ -null set) of a Borel measure  $\eta$  on  $\mathbb{R}$ . (We will also use  $\bar{\eta}$  for the right endpoint of the support). When  $\alpha$  and  $\beta$  are uniform measures, we can compute  $g$  explicitly. For example, if the supports of  $\alpha$  and  $\beta$  are the interval  $[1/2, 3/2]$  then, for  $s, t > 0$ ,

$$g(s, t) = s \log \left( 2 + \frac{t-s + \sqrt{(t-s)^2 + 16st}}{4s} \right) + t \log \left( 2 + \frac{s-t + \sqrt{(s-t)^2 + 16st}}{4t} \right). \quad (1.6)$$

We obtain similar results for the geometric model in which explicit formulas arise when  $\alpha$  and  $\beta$  have densities proportional to  $x \mapsto 1/x$ . We deduce from (1.5) that  $g$  is linear or infinite if  $\underline{\alpha} = \underline{\beta} = 0$ , and is differentiable if  $\underline{\alpha} + \underline{\beta} > 0$ . In the latter case, in contrast with (1.3),  $g$  may be linear close to the axes depending on the behaviors of  $\alpha$  and  $\beta$  near  $\underline{\alpha}$  and  $\underline{\beta}$ . More precisely, there exist constants  $0 \leq c_1 < c_2 \leq \infty$  such that  $g$  is strictly concave only inside the cone  $c_1 < s/t < c_2$ , see Corollary 2.3. This is illustrated through the level set  $g = 1$  in Figure 1.1 below. We expect

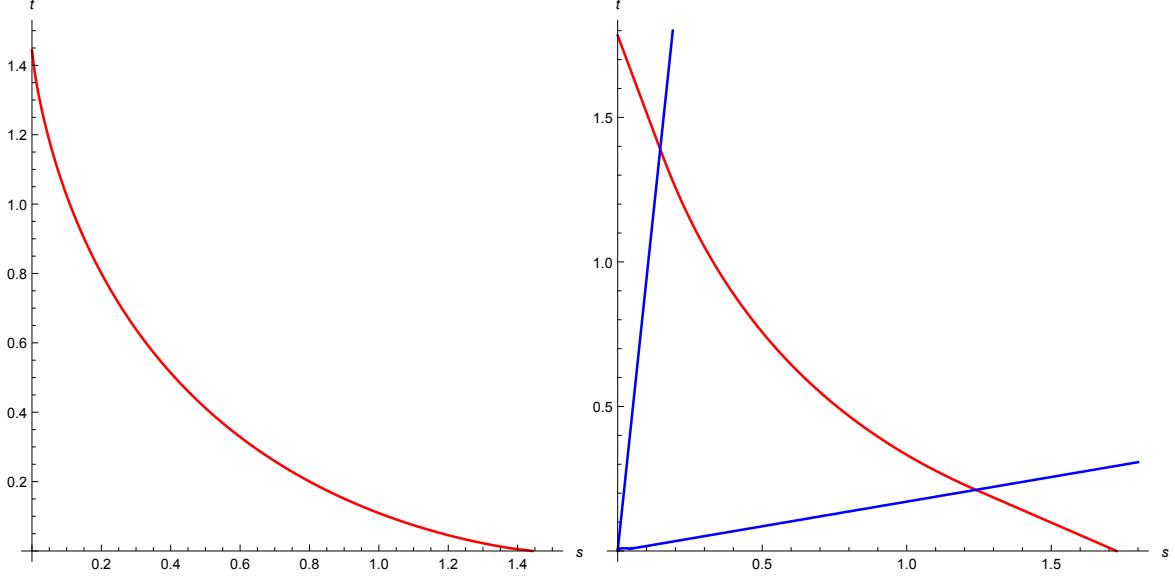


Figure 1.1: The plot of  $g = 1$  for (1.6) where  $c_1 = 0$  and  $c_2 = \infty$  (left). The plot of  $g = 1$  and the rays  $s/t = c_1 = (-8 + 12 \log 2)/3 \approx 0.105922$  and  $s/t = c_2 = 4/(9 - 12 \log 2) \approx 5.863092$  for (1.5) when  $\alpha(dx) = \mathbf{1}_{\{0 \leq x \leq 1\}} 3x^2 dx$  and  $\beta(dx) = \mathbf{1}_{\{1 \leq x \leq 2\}} 4(x-1)^3 dx$  (right).

the finer statistics of  $G(\lfloor ns \rfloor, \lfloor nt \rfloor)$  as  $n \rightarrow \infty$  to be qualitatively different in the linear and concave sectors. This has been confirmed for large deviation properties in [9]. A project currently underway is to understand the limit fluctuations.

A short discussion of the technical aspects of the paper is in order. To calculate the shape function, we rely on certain stationary processes with explicit product-form distributions. This approach dates back to [27] and is illustrated in [29] to derive (1.4), which we briefly outline. Introduce a parameter  $z \in (q, 1/q)$  and boundary weights  $\{W(k, 0), W(0, k) : k \in \mathbb{Z}_+\}$  such that  $\{W(i, j) : i, j \in \mathbb{Z}_+\}$  are independent,  $W(0, 0) = 0$ , and  $W(k, 0)$  and  $W(0, k)$  are geometrically distributed with parameters  $q/z$  and  $qz$ , respectively. Choose the boundary values in recursion (1.1) as  $G(i, 0) = \sum_{k=1}^i W(k, 0)$  and  $G(0, j) = \sum_{k=1}^j W(0, k)$  for  $i, j \in \mathbb{N}$ . Then the resulting model turns out to be *stationary* in the sense that the distributions of the processes  $\{G(i, n) - G(i-1, n) : i \in \mathbb{N}\}$  and  $\{G(n, j) - G(n, j-1) : j \in \mathbb{N}\}$  do not depend on  $n$ . Consequently, these distributions are product measures with geometric marginals. This allows computing the shape function  $g_z$  of the stationary model and relating it to  $g$  via

$$g_z(1, 1) = \sup_{t \in [0, 1]} \max\{g_z(1-t, 0) + g(t, 1), g_z(0, 1-t) + g(1, t)\} \quad (1.7)$$

for  $z \in (q, 1/q)$ . Then (1.4) can be extracted from (1.7). The main observation of the present work is that, given  $(\mathbf{a}, \mathbf{b})$ , if we choose the parameters of  $W(i, 0)$  and  $W(0, j)$  as  $a_i/z$  and  $b_j z$  for  $z \in (\bar{\alpha}, 1/\bar{\beta})$ , we still obtain a stationary model. Then, adapting some arguments from [29], we also arrive at (1.7). To identify  $g$ , [29] uses the symmetry  $g(s, t) = g(t, s)$ , which is not true for the inhomogeneous model unless  $\alpha = \beta$ . For this step, we develop an argument that removes the need for symmetry and makes only a few general assumptions on  $g$  and  $g_z$ .

**Literature review.** We mention briefly some related results and conjectures beginning with the case of i.i.d.  $P$ . For the exponential and geometric models,

$$g(s, t) = m(s + t) + 2\sqrt{\sigma^2 st}, \quad (1.8)$$

where  $m$  and  $\sigma^2$  are the common mean and the variance of the weights. Furthermore,  $g$  satisfies (1.8) up to an error of order  $o(\sqrt{t})$  as  $t \downarrow 0$  provided that the weights have sufficiently light tail [21]. There are also known cases in which  $g$  has linear segments. For example, if the weights are bounded by 1 and  $P(W(i, j) = 1) = p > p_c$ , where  $p_c$  is the critical probability for the oriented

site percolation, then  $g(s, t) = s + t$  in a nontrivial cone [1], [7], [20]. Nevertheless, it is expected that  $g$  is strictly concave and differentiable for a large class of  $P$ , for instance, when the weights have continuous distributions with enough moments. In our setting,  $g$  also enjoys these properties under some moment conditions but can be far from (1.8) as (1.6) exemplifies. More recently, variational formulas in terms of stationary, integrable cocycles have been developed for  $g$  under the mild assumption that the weights have finite  $2 + \epsilon$  moment for some  $\epsilon > 0$  [11]. Minimizers of these formulas are identified as Busemann functions in [10] relying on some fixed point results from queueing theory [19].

There has also been interest to identify the shape function for non-i.i.d.  $P$ . For the exponential model with columnwise inhomogeneity (i.i.d.  $\mathbf{a}$  and constant  $\mathbf{b}$ ), [30] obtained a variational description of  $g$ , which (1.5) includes as a special case. Asymptotics of  $g$  near the axes are determined for more general  $P$  in [18]. Their Theorem 2.4 can be deduced from (1.5). Another direction of generalizing the exponential and geometric models is to choose the parameters at site  $(i, j)$  as  $\lambda = \Lambda(i/n, j/n)$  and  $q = Q(i/n, j/n)$  for some deterministic functions  $\Lambda$  and  $Q$  that encode inhomogeneity. Then, under suitable conditions,  $g$  can be characterized as the unique monotone viscosity solution of a certain Hamilton-Jacobi equation [4].

While we will not take advantage of it in the present paper, we mention that exact solvability of the exponential and geometric models goes beyond the explicit limits (1.3) and (1.4). The distributions of the last-passage times can be expressed as a Fredholm determinant with an explicit kernel. Using this, [12] established that  $G(\lfloor ns \rfloor, \lfloor nt \rfloor)$  has fluctuations of order  $n^{1/3}$  and converges weakly, after suitable rescaling, to the Tracy-Widom GUE distribution. These features are characteristic of the conjectural Kardar-Parisi-Zhang (KPZ) universality class, see survey [6]. More generally, Fredholm determinant representations of the distribution of the last-passage times under  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  have been derived by relating  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  to the Schur measures introduced in [24] and, thereby, to determinantal point processes with explicit correlation kernels [3], [13], [14], [15].

**Outline.** Our results are formally stated in Section 2. We sketch the existence and the basic properties of  $g$  in Section 3. We discuss stationary versions of the exponential and geometric models in Section 4. We prove (1.5) in Section 5.

**Notation and conventions.**  $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ . For  $x \in \mathbb{R}$ , define  $\lfloor x \rfloor = \max\{i \in \mathbb{Z} : i \leq x\}$  and  $\lceil x \rceil = \min\{i \in \mathbb{Z} : i \geq x\}$ . Also,  $x \vee y = \max\{x, y\}$  and  $x \wedge y = \min\{x, y\}$  for  $x, y \in \mathbb{R}$ . The imaginary unit is denoted by  $\mathbf{i}$ . Adjectives *increasing* and *decreasing* are used in the strict sense. For convenience, we set  $1/0 = \infty$  and  $1/\infty = 0$ .

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## 2 Results

Let  $\mathbb{E}$  denote the expectation under  $\mu$  (the distribution of  $(\mathbf{a}, \mathbf{b})$ ). Recall that  $a = a_1$  and  $b = b_1$ . It is convenient to break (1.5) into the next two theorems.

**Theorem 2.1.** *Suppose that  $\underline{\alpha} + \underline{\beta} > 0$  in the exponential model. Then*

$$g(s, t) = \inf_{z \in (-\underline{\alpha}, \underline{\beta})} \left\{ s \mathbb{E} \left[ \frac{1}{a + z} \right] + t \mathbb{E} \left[ \frac{1}{b - z} \right] \right\} \quad \text{for } s, t > 0. \quad (2.1)$$

Hence,  $g$  depends on  $(\mathbf{a}, \mathbf{b})$  only through the marginal distributions  $\alpha$  and  $\beta$ . Let us write  $g^{\alpha, \beta}$  to indicate this. Replacing  $z$  with  $-z$  in (2.1) reveals that  $g^{\alpha, \beta}(s, t) = g^{\beta, \alpha}(t, s)$  for  $s, t > 0$ , which is expected due to the symmetric roles of  $\mathbf{a}$  and  $\mathbf{b}$  in the model. In particular, if  $\alpha$  and  $\beta$  are the same then  $g(s, t) = g(t, s)$  for  $s, t > 0$ . Also, (by dominated convergence) the infimum can be taken over  $[-\underline{\alpha}, \underline{\beta}]$  in (2.1). When  $\underline{\alpha} = \underline{\beta} = 0$ , this interval degenerates to  $\{0\}$  and we expect that  $g(s, t) = s \mathbb{E}[1/a] + t \mathbb{E}[1/b]$  for  $s, t > 0$ . Indeed, this is true.

**Theorem 2.2.** *Suppose that  $\underline{\alpha} = \underline{\beta} = 0$  in the exponential model. Then*

$$g(s, t) = s \mathbb{E} \left[ \frac{1}{a} \right] + t \mathbb{E} \left[ \frac{1}{b} \right] \quad \text{for } s, t > 0.$$

We turn to the concavity and differentiability properties of  $g$ . In the case  $\underline{\alpha} + \underline{\beta} > 0$ , define the critical values  $c_1 = \frac{E[(b + \underline{\alpha})^{-2}]}{E[(a - \underline{\alpha})^{-2}]}$  and  $c_2 = \frac{E[(b - \underline{\beta})^{-2}]}{E[(a + \underline{\beta})^{-2}]}$ . Note that  $0 \leq c_1 < c_2 \leq \infty$ . Also,  $c_1 = 0$  if and only if  $E[(a - \underline{\alpha})^{-2}] = \infty$ , and  $c_2 = \infty$  if and only if  $E[(b - \underline{\beta})^{-2}] = \infty$ .

**Corollary 2.3.** *Suppose that  $\underline{\alpha} + \underline{\beta} > 0$  in the exponential model. Then*

- (a)  $g(s, t) = sE[(a - \underline{\alpha})^{-1}] + tE[(b + \underline{\alpha})^{-1}]$  for  $s/t \leq c_1$ .
- (b)  $g(s, t) = sE[(a + \underline{\beta})^{-1}] + tE[(b - \underline{\beta})^{-1}]$  for  $s/t \geq c_2$ .
- (c)  $g(cs_1 + (1-c)s_2, ct_1 + (1-c)t_2) > cg(s_1, t_1) + (1-c)g(s_2, t_2)$  for  $c \in (0, 1)$  and  $s_1, s_2, t_1, t_2 > 0$  such that  $c_1 < s_1/t_1, s_2/t_2 < c_2$  and  $(s_1, t_1) \neq k(s_2, t_2)$  for any  $k \in \mathbb{R}$ .
- (d)  $g$  is continuously differentiable.

By Schwarz inequality, if  $c_1 > 0$  then  $E[(a - \underline{\alpha})^{-1}] < \infty$  and if  $c_2 < \infty$  then  $E[(b - \underline{\beta})^{-1}] < \infty$ . Hence,  $g$  is finite and linear in  $(s, t)$  in the regions  $s/t \leq c_1$  and  $s/t \geq c_2$ .

*Proof of Corollary 2.3.* Let  $A(z) = E[(a + z)^{-1}]$  for  $z > -\underline{\alpha}$  and  $B(z) = E[(b - z)^{-1}]$  for  $z < \underline{\beta}$ . Using dominated convergence,  $A$  and  $B$  can be differentiated under the expectation. Thus,  $A'(z) = -E[(a + z)^{-2}]$ ,  $B'(z) = E[(b - z)^{-2}]$ ,  $A''(z) = 2E[(a + z)^{-3}]$ ,  $B''(z) = 2E[(b - z)^{-3}]$ , etc. Also, define  $A, B$  and their derivatives at the endpoints by substituting  $-\underline{\alpha}$  and  $\underline{\beta}$  for  $z$  in the preceding formulas. Then, by monotone convergence, the values at the endpoints match the appropriate one-sided limits, that is,  $A(-\underline{\alpha}) = \lim_{z \downarrow -\underline{\alpha}} A(z) = E[(a - \underline{\alpha})^{-1}]$ ,  $B(\underline{\beta}) = \lim_{z \uparrow \underline{\beta}} B(z) = E[(b - \underline{\beta})^{-1}]$ , and similarly for the derivatives.

Since  $A'$  and  $B'$  are increasing and continuous on  $(-\underline{\alpha}, \underline{\beta})$ , the derivative  $z \mapsto sA'(z) + tB'(z)$  is positive if  $s/t \leq c_1$ , is negative if  $s/t \geq c_2$  and has a unique zero if  $c_1 < s/t < c_2$ . Hence, (a) and (b) follow, and if  $c_1 < s/t < c_2$  then  $g(s, t) = sA(z) + tB(z)$ , where  $z \in (-\underline{\alpha}, \underline{\beta})$  is the unique solution of the equation

$$-\frac{B'(z)}{A'(z)} = \frac{s}{t}. \quad (2.2)$$

Since  $-B'/A'$  is increasing and continuous, it has an increasing inverse  $\zeta$  defined on  $(c_1, c_2)$ . Let  $s_1, t_1, s_2, t_2$  be as in (c). Then  $\zeta(s_1/t_1) \neq \zeta(s_2/t_2)$ , which implies the strict inequality

$$\begin{aligned} (s_1 + s_2)A(z) + (t_1 + t_2)B(z) &> s_1A(\zeta(s_1/t_1)) + t_1B(\zeta(s_1/t_1)) \\ &+ s_2A(\zeta(s_2/t_2)) + t_2B(\zeta(s_2/t_2)) \\ &= g(s_1, t_1) + g(s_2, t_2) \end{aligned} \quad (2.3)$$

for any  $z \in (-\underline{\alpha}, \underline{\beta})$ . Note that  $c_1 < (s_1 + s_2)/(t_1 + t_2) < c_2$ . Setting  $z = \zeta((s_1 + s_2)/(t_1 + t_2))$  in (2.3) yields  $g(s_1 + s_2, t_1 + t_2) > g(s_1, t_1) + g(s_2, t_2)$ , and (c) comes from this and homogeneity. Since  $-B'/A'$  is continuously differentiable with positive derivative (as  $A'', B', B > 0$  and  $A' < 0$  on  $(-\underline{\alpha}, \underline{\beta})$ ), by the inverse function theorem,  $\zeta$  is continuously differentiable as well. Using (2.2), we compute the gradient of  $g$  for  $c_1 < s/t < c_2$  as  $\nabla g(s, t) = (A(\zeta(s/t)), B(\zeta(s/t)))$ , which tends to  $(A(-\underline{\alpha}), B(-\underline{\alpha}))$  as  $s/t \rightarrow c_1$  and to  $(A(\underline{\beta}), B(\underline{\beta}))$  as  $s/t \rightarrow c_2$ . Hence, (d).  $\square$

When  $\alpha$  and  $\beta$  are uniform distributions, we can compute the infimum in (2.1) explicitly.

**Corollary 2.4.** *Let  $\lambda, l, m > 0$ . Suppose that  $\alpha$  and  $\beta$  are uniform distributions on  $[\lambda/2, \lambda/2 + l]$  and  $[\lambda/2, \lambda/2 + m]$ , respectively. Then, for  $s, t > 0$ ,*

$$\begin{aligned} g(s, t) &= \frac{s}{l} \log \left( 1 + \frac{l}{\lambda} + \frac{l}{\lambda} \cdot \frac{lt - ms + \sqrt{(lt - ms)^2 + 4st(\lambda + l)(\lambda + m)}}{2s(\lambda + m)} \right) \\ &+ \frac{t}{m} \log \left( 1 + \frac{m}{\lambda} + \frac{m}{\lambda} \cdot \frac{ms - lt + \sqrt{(lt - ms)^2 + 4st(\lambda + l)(\lambda + m)}}{2t(\lambda + l)} \right). \end{aligned}$$

*Proof.* Since  $\alpha$  and  $\beta$  are uniform distributions,

$$A(z) = \mathbb{E} \left[ \frac{1}{a+z} \right] = \frac{1}{l} \log \left( 1 + \frac{l}{z + \lambda/2} \right) \quad B(z) = \mathbb{E} \left[ \frac{1}{b-z} \right] = \frac{1}{m} \log \left( 1 + \frac{m}{-z + \lambda/2} \right)$$

for  $z \in (-\lambda/2, \lambda/2)$ . We compute the derivatives as

$$A'(z) = -\frac{1}{(z + \lambda/2)(z + \lambda/2 + l)} \quad B'(z) = \frac{1}{(-z + \lambda/2)(-z + \lambda/2 + m)}.$$

Because  $A'(-\lambda/2) = -\infty$  and  $B'(\lambda/2) = \infty$ , we have  $c_1 = 0$  and  $c_2 = \infty$ . Also, (2.2) leads to

$$(s-t)z^2 - (s(\lambda+m) + t(\lambda+l))z + s\lambda(\lambda+2m)/4 - t\lambda(\lambda+2l)/4 = 0.$$

It follows from the discriminant formula that the solution in the interval  $(-\lambda/2, \lambda/2)$  is

$$z = \frac{\lambda}{2} \frac{s(\lambda+2m) - t(\lambda+2l)}{s(\lambda+m) + t(\lambda+l) + \sqrt{(sm+tl)^2 + 4st\lambda(\lambda+m+l)}}.$$

Inserting this into  $g(s, t) = sA(z) + tB(z)$  and some elementary algebra yield the result.  $\square$

The preceding argument can be repeated when  $l = 0$  or  $m = 0$ . In these cases,  $\alpha$  and  $\beta$  are understood as point masses at  $\lambda/2$ . For instance, when  $l = 0$  and  $m > 0$ , we obtain

$$g(s, t) = \frac{2s\lambda + ms + \sqrt{(ms)^2 + 4st\lambda(\lambda+m)}}{2\lambda(\lambda+m)} + \frac{t}{m} \log \left( 1 + \frac{m}{\lambda} + \frac{m}{\lambda} \cdot \frac{ms + \sqrt{(ms)^2 + 4st\lambda(\lambda+m)}}{2t\lambda} \right)$$

When  $l = 0$  and  $m = 0$ , we recover (1.3).

We can also determine  $g$  along the diagonal when  $\alpha$  and  $\beta$  are the same.

**Corollary 2.5.** *Suppose that  $\alpha = \beta$ . Then  $g(s, s) = 2s \mathbb{E} \left[ \frac{1}{a} \right]$  for  $s > 0$ .*

*Proof.* We have  $(a+z)^{-1} + (a-z)^{-1} \geq 2a^{-1}$  for  $|z| \leq \alpha$  with equality if only if  $z = 0$ . Therefore,

$$g(s, s) = s \inf_{z \in (-\alpha, \alpha)} \mathbb{E} \left[ \frac{1}{a+z} + \frac{1}{a-z} \right] = 2s \mathbb{E} \left[ \frac{1}{a} \right]. \quad \square$$

We only report the analogous results for the geometric model.

**Theorem 2.6.** *Suppose that  $\bar{\alpha}\bar{\beta} < 1$  in the geometric model. Then*

$$g(s, t) = \inf_{z \in (\bar{\alpha}, 1/\bar{\beta})} \left\{ s \mathbb{E} \left[ \frac{a/z}{1-a/z} \right] + t \mathbb{E} \left[ \frac{bz}{1-bz} \right] \right\} \quad \text{for } s, t > 0.$$

**Theorem 2.7.** *Suppose that  $\bar{\alpha} = \bar{\beta} = 1$  in the geometric model. Then*

$$g(s, t) = s \mathbb{E} \left[ \frac{a}{1-a} \right] + t \mathbb{E} \left[ \frac{b}{1-b} \right] \quad \text{for } s, t > 0.$$

**Corollary 2.8.** *Let  $q \in (0, 1)$  and  $0 < l, m < \sqrt{q}$ . Choose  $\alpha$  and  $\beta$  as the distributions with densities proportional to  $x \mapsto 1/x$  on the intervals  $[\sqrt{q}-l, \sqrt{q}]$  and  $[\sqrt{q}-m, \sqrt{q}]$ , respectively. Then*

$$g(s, t) = \frac{s}{L} \log \left( 1 + \frac{l\sqrt{q}}{1-q} + \frac{l}{1-q} \frac{ly - mx + \sqrt{(ly - mx)^2 + 4xy(1+m\sqrt{q}-q)(1+l\sqrt{q}-q)}}{2x(1+m\sqrt{q}-q)} \right) \\ + \frac{t}{M} \log \left( 1 + \frac{m\sqrt{q}}{1-q} + \frac{m}{1-q} \frac{mx - ly + \sqrt{(ly - mx)^2 + 4xy(1+m\sqrt{q}-q)(1+l\sqrt{q}-q)}}{2y(1+l\sqrt{q}-q)} \right)$$

for  $s, t > 0$ , where  $x = slM$ ,  $y = tmL$ ,  $L = \log \left( \frac{\sqrt{q}}{\sqrt{q}-l} \right)$  and  $M = \log \left( \frac{\sqrt{q}}{\sqrt{q}-m} \right)$ .

**Corollary 2.9.** *Suppose that  $\alpha = \beta$ . Then  $g(s, s) = 2s \mathbb{E} \left[ \frac{a}{1-a} \right]$  for  $s > 0$ .*

### 3 The existence of the shape function

**Lemma 3.1.** *There exists a deterministic function  $g : (0, \infty)^2 \rightarrow [0, \infty]$  such that*

$$\lim_{n \rightarrow \infty} \frac{G(\lfloor ns \rfloor, \lfloor nt \rfloor)}{n} = g(s, t) \quad \text{for } s, t > 0 \quad \mathbb{P}\text{-a.s.}$$

Furthermore,  $g$  is nondecreasing, homogeneous and concave.

Here, nondecreasing means that  $g(s', t') \leq g(s, t)$  for  $0 < s' \leq s$  and  $0 < t' \leq t$ , and homogeneity means that  $g(cs, ct) = cg(s, t)$  for  $s, t, c > 0$ . In the exponential model,  $g$  is finite if  $\underline{\alpha} + \underline{\beta} > 0$ . This is by the standard properties of the stochastic order [31, Theorem 1.A3]. Briefly, the i.i.d. measure  $P$  on  $\mathbb{R}_+^{\mathbb{N}^2}$  under which each  $W(i, j)$  is exponentially distributed with rate  $\underline{\alpha} + \underline{\beta}$  stochastically dominates  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  and  $G$  is a nondecreasing function of the weights. Thus,  $g(s, t)$  does not exceed the right-hand side of (1.3) with  $\lambda = \underline{\alpha} + \underline{\beta}$ . Similarly,  $g$  is finite in the geometric model if  $\bar{\alpha}\bar{\beta} < 1$ . Extend  $g$  to  $\mathbb{R}_+^2$  by setting  $g(0, 0) = 0$ ,  $g(s, 0) = \lim_{t \downarrow 0} g(s, t)$  and  $g(0, t) = \lim_{s \downarrow 0} g(s, t)$  for  $s, t > 0$ .

Lemma 3.1 can be proved using the ergodicity properties of  $\mathbb{P}$  and superadditivity of the last-passage times. As this is quite standard, we will leave out many details. For  $k, l \in \mathbb{Z}_+$ , let  $\theta_{k,l} : \mathbb{R}_+^{\mathbb{N}^2} \rightarrow \mathbb{R}_+^{\mathbb{N}^2}$  be given by  $\theta_{k,l}(\omega)(i, j) = \omega(i + k, j + l)$  for  $i, j \in \mathbb{N}$  and  $\omega \in \mathbb{R}_+^{\mathbb{N}^2}$ . Note that  $\mathbb{P}$  is stationary with respect to  $\theta_{k,l}$  because  $\mathbb{P}(\theta_{k,l}^{-1}(B)) = \mathbb{E} \mathbf{P}_{\mathbf{a}, \mathbf{b}}(\theta_{k,l}^{-1}(B)) = \mathbb{E} \mathbf{P}_{\tau_k(\mathbf{a}), \tau_l(\mathbf{b})}(B) = \mathbb{P}(B)$  for any Borel set  $B \subset \mathbb{R}_+^{\mathbb{N}^2}$ .

**Lemma 3.2.**  *$\mathbb{P}$  is ergodic with respect to  $\theta_{k,l}$  for any  $k, l \in \mathbb{N}$ .*

*Proof.* Suppose  $\theta_{k,l}^{-1}(B) = B$  for some Borel set  $B \subset \mathbb{R}_+^{\mathbb{N}^2}$ . For  $n \geq 1$ , let  $\mathcal{T}_n$  denote the  $\sigma$ -algebra generated by  $A_n$ , the collection of  $W(i, j)$  with  $i > k(n-1)$  and  $j > l(n-1)$ . Then  $B$  is in  $\mathcal{T} = \bigcap_{n \in \mathbb{N}} \mathcal{T}_n$ . Also,  $\mathcal{T}$  is the tail  $\sigma$ -algebra of the  $\sigma$ -algebras generated by  $A_n \setminus A_{n+1}$ . Because  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  is a product measure, by Kolmogorov's 0-1 law,  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}(B) \in \{0, 1\}$ . Therefore,  $\mathbb{P}(B) = \mu(\mathbf{P}_{\mathbf{a}, \mathbf{b}}(B) = 1)$ . On the other hand,

$$(\tau_k \times \tau_l)^{-1}\{\mathbf{P}_{\mathbf{a}, \mathbf{b}}(B) = 1\} = \{\mathbf{P}_{\tau_k \mathbf{a}, \tau_l \mathbf{b}}(B) = 1\} = \{\mathbf{P}_{\mathbf{a}, \mathbf{b}}(\theta_{k,l}^{-1}(B)) = 1\} = \{\mathbf{P}_{\mathbf{a}, \mathbf{b}}(B) = 1\}.$$

Since  $\mu$  is ergodic under  $\tau_k \times \tau_l$ , we conclude that  $\mathbb{P}(B) \in \{0, 1\}$ .  $\square$

*Proof of Lemma 3.1.* Fix  $s, t \in \mathbb{N}$  and define, for integers  $0 \leq m < n$ ,

$$Z(m, n) = -G((n-m)s, (n-m)t) \circ \theta_{ms, mt} = \max_{\pi \in \Pi_{ms+1, mt+1, ns, nt}} \sum_{(i,j) \in \pi} W(i, j).$$

Using the definition and Lemma 3.2, we observe that  $\{Z(m, n) : 0 \leq m < n\}$  is a subadditive process that satisfies the hypotheses of Liggett's subadditive ergodic theorem [17]. Hence,  $Z(0, n)/n = G(ns, nt)/n$  converges  $\mathbb{P}$ -a.s. to a deterministic limit,  $g(s, t)$ . The existence of the limit for all  $s, t > 0$   $\mathbb{P}$ -a.s. and the claimed properties of  $g$  follow as in the case of i.i.d. weights [29, Theorem 2.1].  $\square$

### 4 Stationary distributions of the last-passage increments

Let us extend the sample space to  $\mathbb{R}_+^{\mathbb{Z}_+^2}$ . Now  $W(i, j)$  denotes the projection onto coordinate  $(i, j)$  for  $i, j \in \mathbb{Z}_+$ . Define the last-passage time  $\hat{G}(i, j)$  through recursion (1.1) but with the boundary values  $\hat{G}(i, 0) = \sum_{k=1}^i W(k, 0)$  and  $\hat{G}(0, j) = \sum_{k=1}^j W(0, k)$  for  $i, j \in \mathbb{N}$ . We then have

$$\hat{G}(m, n) = \max_{\pi \in \Pi_{0,0,m,n}} \sum_{(i,j) \in \pi} W(i, j) \quad \text{for } m, n \in \mathbb{Z}_+. \quad (4.1)$$

In the exponential model, for each value of  $(\mathbf{a}, \mathbf{b})$  such that  $a_n \geq \underline{\alpha}$  and  $b_n \geq \underline{\beta}$  for  $n \in \mathbb{N}$  (which holds  $\mu$ -a.s.) and parameter  $z \in (-\underline{\alpha}, \underline{\beta})$ , define  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$  as the product measure on  $\mathbb{R}_+^{\mathbb{Z}_+^2}$  by

$$\begin{aligned} \mathbf{P}_{\mathbf{a}, \mathbf{b}}^z(W(i, j) \geq x) &= \exp(-(a_i + b_j)x) & \mathbf{P}_{\mathbf{a}, \mathbf{b}}^z(W(0, 0) = 0) &= 1 \\ \mathbf{P}_{\mathbf{a}, \mathbf{b}}^z(W(i, 0) \geq x) &= \exp(-(a_i + z)x) & \mathbf{P}_{\mathbf{a}, \mathbf{b}}^z(W(0, j) \geq x) &= \exp(-(b_j - z)x) \end{aligned} \quad (4.2)$$

for  $x \geq 0$  and  $i, j \in \mathbb{N}$ . When  $\underline{\alpha} = \underline{\beta} = 0$ , we make definition (4.2) for  $z = 0$ . Note that the projection of  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$  onto coordinates  $\mathbb{N}^2$  is  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$ . For the geometric model, the construction is similar. For  $z \in (\bar{\alpha}, 1/\bar{\beta})$  and each value of  $(\mathbf{a}, \mathbf{b})$  such that  $a_n \leq \bar{\alpha}$  and  $b_n \leq \bar{\beta}$  for  $n \in \mathbb{N}$ , the measure  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$  is given by

$$\begin{aligned} \mathbf{P}_{\mathbf{a}, \mathbf{b}}^z(W(i, j) \geq k) &= a_i^k b_j^k & \mathbf{P}_{\mathbf{a}, \mathbf{b}}^z(W(0, 0) = 0) &= 1 \\ \mathbf{P}_{\mathbf{a}, \mathbf{b}}^z(W(i, 0) \geq k) &= a_i^k / z^k & \mathbf{P}_{\mathbf{a}, \mathbf{b}}^z(W(0, j) \geq k) &= b_j^k z^k \end{aligned} \quad (4.3)$$

for  $k \in \mathbb{Z}_+$  and  $i, j \in \mathbb{N}$ . When  $\bar{\alpha} = \bar{\beta} = 1$ , definition (4.3) makes sense for  $z = 1$ .

Introduce the increment variables as  $I(m, n) = \hat{G}(m, n) - \hat{G}(m-1, n)$  for  $m \geq 1$  and  $n \geq 0$ , and  $J(m, n) = \hat{G}(m, n) - \hat{G}(m, n-1)$  for  $m \geq 0$  and  $n \geq 1$ . We capture the stationarity of the increments in the following proposition.

**Proposition 4.1.** *Let  $k, l \in \mathbb{Z}_+$ . Under  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$ ,*

- (a)  $I(i, l)$  has the same distribution as  $W(i, 0)$  for  $i \in \mathbb{N}$ .
- (b)  $J(k, j)$  has the same distribution as  $W(0, j)$  for  $j \in \mathbb{N}$ .
- (c) The random variables  $\{I(i, l) : i > k\} \cup \{J(k, j) : j > l\}$  are (jointly) independent.

(1.1) leads to the recursion [29, (2.21)]

$$\begin{aligned} I(m, n) &= I(m, n-1) - I(m, n-1) \wedge J(m-1, n) + W(m, n) \\ J(m, n) &= J(m-1, n) - I(n, n-1) \wedge J(m-1, n) + W(m, n) \end{aligned} \quad (4.4)$$

for  $m, n \in \mathbb{N}$ . Proposition 4.1 can be proved via induction using (4.4) and Lemma 4.2 below. We will omit the induction argument as it is the same as in [29, Theorem 2.4].

**Lemma 4.2.** *Let  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  denote the map  $(x, y, z) \mapsto (x - x \wedge y + z, y - x \wedge y + z, x \wedge y)$ . Let  $P$  be a product measure on  $\mathbb{R}^3$  with marginals  $P_1, P_2, P_3$ . Suppose that one of the following holds.*

- (i)  $P_1, P_2$  and  $P_3$  are exponential distributions with rates  $a, b$  and  $a+b$ , for some  $a, b \in (0, \infty)$ .
- (ii)  $P_1, P_2$  and  $P_3$  are geometric distributions with parameters  $a, b$  and  $ab$ , for some  $a, b \in (0, 1)$ .

*Then  $P(F^{-1}(B)) = P(B)$  for any Borel set  $B \subset \mathbb{R}^3$ .*

In earlier work [29, Lemma 2.3] and [2, Lemma 4.1], Lemma 4.2 was proved by comparing the Laplace transforms of the measures  $P$  and  $P(F^{-1}(\cdot))$ . We include another proof below.

*Proof of Lemma 4.2.* We prove (i) only as the proof of (ii) is the discrete version of the same argument and is simpler. Observe that  $F$  is a bijection on  $\mathbb{R}^3$  with  $F^{-1} = F$ . It suffices to verify the claim for any open set  $B$  in  $\mathbb{R}^3$ . By continuity,  $F^{-1}(B)$  is also open. Furthermore,  $F$  is differentiable on the open set  $\{(x, y, z) : x > y \text{ or } x < y\}$  and its Jacobian equals 1 in absolute value. Hence, by the change of variables [26, Theorem 7.26],

$$\begin{aligned} P(F^{-1}(B)) &= ab(a+b) \int_{F^{-1}(B)} e^{-ax-by-(a+b)z} dx dy dz \\ &= ab(a+b) \int_{F^{-1}(B)} e^{-a(x-x \wedge y+z)-b(y-x \wedge y+z)-(a+b)(x \wedge y)} dx dy dz \\ &= ab(a+b) \int_B e^{-au-bv-(a+b)w} du dv dw = P(B). \quad \square \end{aligned}$$

In the exponential and geometric models, respectively, define

$$\begin{aligned} g_z(s, t) &= s \mathbb{E} \left[ \frac{1}{a+z} \right] + t \mathbb{E} \left[ \frac{1}{b-z} \right] \quad \text{for } s, t \geq 0 \text{ and } z \in [-\alpha, \beta] \\ g_z(s, t) &= s \mathbb{E} \left[ \frac{a/z}{1-a/z} \right] + t \mathbb{E} \left[ \frac{bz}{1-bz} \right] \quad \text{for } s, t \geq 0 \text{ and } z \in [\bar{\alpha}, 1/\bar{\beta}]. \end{aligned}$$



**Lemma 4.3.** *In the exponential model, let  $z \in (-\underline{\alpha}, \underline{\beta})$  if  $\underline{\alpha} + \underline{\beta} > 0$ , and let  $z = 0$  and assume that  $E[1/a + 1/b] < \infty$  if  $\underline{\alpha} = \underline{\beta} = 0$ . In the geometric model, let  $z \in (\bar{\alpha}, 1/\bar{\beta})$  if  $\bar{\alpha}\bar{\beta} < 1$ , and let  $z = 1$  and assume that  $E[a/(1-a) + b/(1-b)] < \infty$  if  $\bar{\alpha} = \bar{\beta} = 1$ . Then*

$$\lim_{n \rightarrow \infty} \frac{\hat{G}(\lfloor ns \rfloor, \lfloor nt \rfloor)}{n} = g_z(s, t) \quad \text{for } s, t \geq 0 \text{ in } \mathbf{P}_{\mathbf{a}, \mathbf{b}}^z \text{-probability for } \mu\text{-a.e. } (\mathbf{a}, \mathbf{b}). \quad (4.5)$$

In fact, the convergence in (4.5) is  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$ -a.s. for  $\mu$ -a.e.  $(\mathbf{a}, \mathbf{b})$  provided that  $\underline{\alpha} + \underline{\beta} > 0$  in the exponential model and  $\bar{\alpha}\bar{\beta} < 1$  in the geometric model [8, Theorem 4.3]. By (1.2), (4.1) and nonnegativity of weights,  $G(m, n) \leq \hat{G}(m, n)$  for  $m, n \in \mathbb{N}$ . Then Lemma 4.3 implies that  $g(s, t) \leq g_z(s, t)$  for any  $s, t \geq 0$ . The main result of this paper is that  $g(s, t) = \inf_z g_z(s, t)$ .

*Proof of Lemma 4.3.* We will consider the exponential model only, the geometric model is treated similarly. Note that  $\hat{G}(\lfloor ns \rfloor, \lfloor nt \rfloor) = \sum_{i=1}^{\lfloor ns \rfloor} I(i, 0) + \sum_{j=1}^{\lfloor nt \rfloor} J(\lfloor ns \rfloor, j)$  for  $s, t \geq 0$  and  $n \in \mathbb{N}$ . By Proposition 4.1,  $\{J(\lfloor ns \rfloor, j) : j \in \mathbb{N}\}$  has the same distribution as  $\{J(0, j) : j \in \mathbb{N}\}$  under  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$ . Hence, it suffices to show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n I(i, 0) = E \left[ \frac{1}{a+z} \right] \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n J(0, j) = E \left[ \frac{1}{b-z} \right]$$

in  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$  for  $\mu$ -a.s. We will only derive the first limit above, for which we will show that, for  $z > -\underline{\alpha}$  and for  $z = -\underline{\alpha}$  when  $E[(a - \underline{\alpha})^{-1}] < \infty$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n I(i, 0) = E \left[ \frac{1}{a+z} \right] \quad \text{in } \mathbf{Q}_{\mathbf{a}}^z \text{ } \mu\text{-a.s.}, \quad (4.6)$$

where  $\mathbf{Q}_{\mathbf{a}}^z$  is the product measure on the coordinates  $\mathbb{N} \times \{0\}$  given by  $\mathbf{Q}_{\mathbf{a}}^z(W(i, 0) \geq x) = e^{-(a_i+z)x}$  for  $i \in \mathbb{N}$  and  $x \geq 0$ . It suffices to prove the convergence in distribution under  $\mathbf{Q}_{\mathbf{a}}^z$   $\mu$ -a.s. because the limit is deterministic.

The characteristic function of  $n^{-1} \sum_{i=1}^n I(i, 0)$  under  $\mathbf{Q}_{\mathbf{a}}^z$  is given by

$$\prod_{i=1}^n \left( 1 - \frac{\mathbf{i}x}{n(a_i + z)} \right)^{-1} = \exp \left( - \sum_{i=1}^n \log \left( 1 - \frac{\mathbf{i}x}{n(a_i + z)} \right) \right) \quad \text{for } x \in \mathbb{R},$$

where the complex logarithm denotes the principal branch. Hence, (4.6) follows if we prove

$$\lim_{n \rightarrow \infty} - \sum_{i=1}^n \log \left( 1 - \frac{\mathbf{i}x}{n(a_i + z)} \right) = \mathbf{i}x E \left[ \frac{1}{a+z} \right] \quad \text{for } x \in \mathbb{R} \quad \mu\text{-a.s.},$$

Using the bound  $|\log(1 + \mathbf{i}x)| \leq |x|$  for  $x \in \mathbb{R}$  and the ergodicity of  $\mathbf{a}$ , we obtain

$$\limsup_{n \rightarrow \infty} \left| \sum_{i=1}^n \log \left( 1 - \frac{\mathbf{i}x}{n(a_i + z)} \right) \right| \leq \lim_{n \rightarrow \infty} \frac{|x|}{n} \sum_{i=1}^n \frac{1}{a_i + z} = |x| E \left[ \frac{1}{a+z} \right] \quad \text{for } x \in \mathbb{R} \quad \mu\text{-a.s.}$$

Therefore, it suffices to prove the following for  $x \in \mathbb{R}$   $\mu$ -a.s.

$$\lim_{n \rightarrow \infty} - \sum_{i=1}^n \arg \left( 1 - \frac{\mathbf{i}x}{n(a_i + z)} \right) - x E \left[ \frac{1}{a+z} \right] = \lim_{n \rightarrow \infty} \sum_{i=1}^n \arctan \left( \frac{x}{n(a_i + z)} \right) - \frac{x}{n(a_i + z)} = 0. \quad (4.7)$$

Since  $\arctan x = \int_0^x (1 + u^2)^{-1} du$ , we can rewrite the second sum above as

$$\sum_{i=1}^n \int_0^{xn^{-1}(a_i+z)^{-1}} \frac{du}{1+u^2} - \frac{x}{n(a_i+z)} = - \sum_{i=1}^n \int_0^{xn^{-1}(a_i+z)^{-1}} \frac{u^2 du}{1+u^2} = - \frac{x}{n} \sum_{i=1}^n \int_0^{(a_i+z)^{-1}} \frac{x^2 v^2 dv}{n^2 + x^2 v^2},$$

where we changed the variables via  $u = vx/n$ . Pick  $M > 0$ . The limsup as  $n \rightarrow \infty$  of the absolute value of the last sum is bounded  $\mu$ -a.s. by  $|x|$  times

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \int_0^{(a_i+z)^{-1}} \frac{x^2 v^2 dv}{M^2 + x^2 v^2} = \mathbb{E} \left[ \int_0^{(a+z)^{-1}} \frac{x^2 v^2 dv}{M^2 + x^2 v^2} \right],$$

where the a.s. convergence is due to the ergodicity of  $\mathbf{a}$  and the integrability of

$$\int_0^{(a+z)^{-1}} \frac{x^2 v^2 dv}{M^2 + x^2 v^2} \leq \frac{1}{a+z}.$$

The last integral is monotone in  $x^2$  and vanishes as  $M \rightarrow \infty$ . Hence, (4.7) holds for  $x \in \mathbb{R}$   $\mu$ -a.s.  $\square$

The next proposition relates  $g_z$  to  $g$  through a variational formula.

**Proposition 4.4.**

$$g_z(1, 1) = \sup_{t \in [0, 1]} \max\{g_z(1-t, 0) + g(t, 1), g_z(0, 1-t) + g(1, t)\} \quad (4.8)$$

For  $z \in (-\underline{\alpha}, \underline{\beta})$  in the exponential model and for  $z \in (\bar{\alpha}, 1/\bar{\beta})$  in the geometric model.

*Proof.* Fix  $z \in (-\underline{\alpha}, \underline{\beta})$  in the exponential model. Since  $g \leq g_z$  and  $g_z$  is linear, (4.8) with  $\geq$  instead of  $=$  is immediate. For the opposite inequality, we adapt the argument in [29, Proposition 2.7]. It follows from (1.2) and (4.1) that

$$\hat{G}(n, n) = \max_{k \in [n]} \max\{\hat{G}(k, 0) + G(n-k+1, n) \circ \theta_{k-1, 0}, \hat{G}(0, k) + G(n, n-k+1) \circ \theta_{0, k-1}\}. \quad (4.9)$$

Let  $L \in \mathbb{N}$  and consider  $n > L$  large enough so that  $\lceil (i+1)n/L \rceil > \lceil in/L \rceil$  for  $0 \leq i < L$ . For any  $k \in [n]$  there exists some  $0 \leq i < L$  such that  $\lceil in/L \rceil < k \leq \lceil (i+1)n/L \rceil$ , and the weights are nonnegative. Therefore, (4.9) implies that

$$\begin{aligned} \hat{G}(n, n) &\leq \max_{0 \leq i < L} \max\{\hat{G}(\lceil (i+1)n/L \rceil, 0) + G(\lfloor (1-i/L)n \rfloor, n) \circ \theta_{\lceil in/L \rceil, 0}, \\ &\quad \hat{G}(0, \lceil (i+1)n/L \rceil) + G(n, \lfloor (1-i/L)n \rfloor) \circ \theta_{0, \lceil in/L \rceil}\}. \end{aligned} \quad (4.10)$$

By stationarity of  $\mathbb{P}$ , we have the following limits in  $\mathbb{P}$ -probability.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{G(\lfloor (1-i/L)n \rfloor, n) \circ \theta_{\lceil in/L \rceil, 0}}{n} &= g(1-i/L, 1) \\ \lim_{n \rightarrow \infty} \frac{G(n, \lfloor (1-i/L)n \rfloor) \circ \theta_{0, \lceil in/L \rceil}}{n} &= g(1, 1-i/L) \end{aligned} \quad (4.11)$$

Hence, these limits are  $\mathbb{P}$ -a.s. and, consequently,  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  a.s.  $\mu$ -a.s. if  $n \rightarrow \infty$  along a suitable sequence  $(n_k)_{k \in \mathbb{N}}$ . Also, by Lemma 4.3, there is a subsequence  $(n'_k)_{k \in \mathbb{N}}$  in  $\mathbb{N}$   $\mu$ -a.s. such that  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$  a.s.

$$\lim_{k \rightarrow \infty} \frac{\hat{G}(\lceil (i+1)n'_k/L \rceil, 0)}{n'_k} = g_z(i+1/L, 0) \quad \lim_{k \rightarrow \infty} \frac{\hat{G}(0, \lceil (i+1)n'_k/L \rceil)}{n'_k} = g_z(0, i+1/L) \quad (4.12)$$

Because  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  is a projection of  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$ , we can choose  $(\mathbf{a}, \mathbf{b})$  such that (4.11) and (4.12) hold  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}^z$ -a.s. Hence, we obtain from (4.10) that

$$\begin{aligned} g_z(1, 1) &\leq \max_{0 \leq i < L} \max\{g_z((i+1)/L, 0) + g(1-i/L, 1), g_z(0, (i+1)/L) + g(1, 1-i/L)\} \\ &\leq \sup_{0 \leq t \leq 1} \max\{g(t, 1) + g_z(1-t, 0), g(1, t) + g_z(0, 1-t)\} + \frac{\mathbb{E}[(a+z)^{-1}] + \mathbb{E}[(b-z)^{-1}]}{L}. \end{aligned}$$

Finally, let  $L \rightarrow \infty$ . The geometric model is treated similarly.  $\square$

## 5 Variational characterization of the shape function

We now prove Theorems 2.1 and 2.2. The assumption  $\underline{\alpha} + \underline{\beta} > 0$  is in force until the proof of Theorem 2.2. We begin with computing  $g$  on the boundary. Recall that  $g$  is extended to the boundary of  $\mathbb{R}_+^2$  through limits. By homogeneity, it suffices to determine  $g(1, 0)$  and  $g(0, 1)$ .

**Lemma 5.1.**

$$g(1, 0) = \mathbb{E} \left[ \frac{1}{a + \underline{\beta}} \right] \quad g(0, 1) = \mathbb{E} \left[ \frac{1}{b + \underline{\alpha}} \right].$$

*Proof.* We have  $g(1, 0) \leq g_z(1, 0) = \mathbb{E}[(a + z)^{-1}]$  for all  $z \in (-\underline{\alpha}, \underline{\beta})$ . Letting  $z \uparrow \underline{\beta}$  yields the upper bound  $g(1, 0) \leq \mathbb{E}[(a + \underline{\beta})^{-1}]$ . Now the lower bound. Let  $\epsilon > 0$ . By Lemma 3.1, (4.6) and since  $\mu(b_1 \leq \underline{\beta} + \epsilon) > 0$ , there exists  $(\mathbf{a}, \mathbf{b})$  such that  $b_1 \leq \underline{\beta} + \epsilon$  and

$$\lim_{n \rightarrow \infty} \frac{G(n, \lfloor n\epsilon \rfloor)}{n} = g(1, \epsilon) \quad \mathbf{P}_{\mathbf{a}, \mathbf{b}} \text{-a.s.} \quad (5.1)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n I(i, 0) = \mathbb{E} \left[ \frac{1}{a + \underline{\beta} + \epsilon} \right] \quad \text{in } \mathbf{Q}_{\mathbf{a}}^{\beta + \epsilon} \text{-probability.} \quad (5.2)$$

( $\mathbf{Q}_{\mathbf{a}}^z$  is defined immediately after (4.6)). The distribution of  $\{W(i, 1) : 1 \leq i \leq n\}$  under  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  stochastically dominates the distribution of  $\{I(i, 0) : 1 \leq i \leq n\}$  under  $\mathbf{Q}_{\mathbf{a}}^{\beta + \epsilon}$  as these distributions have product forms and  $i$ th marginals are exponentials with rates  $a_i + b_1 \leq a_i + \underline{\beta} + \epsilon$  for  $i \in [n]$ . Therefore, for  $x \in \mathbb{R}$  and  $n \geq 1/\epsilon$ ,

$$\mathbf{P}_{\mathbf{a}, \mathbf{b}}(G(n, \lfloor n\epsilon \rfloor) \geq nx) \geq \mathbf{P}_{\mathbf{a}, \mathbf{b}} \left( \sum_{i=1}^n W(i, 1) \geq nx \right) \geq \mathbf{Q}_{\mathbf{a}}^{\beta + \epsilon} \left( \sum_{i=1}^n I(i, 0) \geq nx \right).$$

Set  $x = \mathbb{E}[(a + \underline{\beta} + \epsilon)^{-1}] - \epsilon$  and let  $n \rightarrow \infty$ . By (5.1) and (5.2), we obtain  $g(1, \epsilon) \geq x$ . Sending  $\epsilon \downarrow 0$  gives  $g(1, 0) \geq \mathbb{E}[(a + \underline{\beta})^{-1}]$ . Computation of  $g(0, 1)$  is similar.  $\square$

We now extract  $g$  from (1.7). For this, we will only use the boundary values of  $g$  provided in Lemma 5.1, and that  $A(z) = \mathbb{E}[(a + z)^{-1}]$  and  $B(z) = \mathbb{E}[(b - z)^{-1}]$  are continuous, strictly monotone functions on  $(-\underline{\alpha}, \underline{\beta})$ .

**Lemma 5.2.** *Let  $r$  be a positive, continuous function on  $[0, \pi/2]$ . For  $z \in (-\underline{\alpha}, \underline{\beta})$ ,*

$$\sup_{0 \leq \theta \leq \pi/2} \{g(x(\theta), y(\theta)) - g_z(x(\theta), y(\theta))\} = 0,$$

where  $(x(\theta), y(\theta)) = (r(\theta) \cos \theta, r(\theta) \sin \theta)$  for  $0 \leq \theta \leq \pi/2$ .

*Proof.* We can rewrite (4.8) as

$$\begin{aligned} A(z) + B(z) &= \sup_{\pi/4 \leq \theta \leq \pi/2} \{(1 - \cot \theta)A(z) + g(\cot \theta, 1)\} \vee \sup_{0 \leq \theta \leq \pi/4} \{(1 - \tan \theta)B(z) + g(1, \tan \theta)\} \\ &= \sup_{\pi/4 \leq \theta \leq \pi/2} \left\{ \left(1 - \frac{x(\theta)}{y(\theta)}\right) A(z) + g\left(\frac{x(\theta)}{y(\theta)}, 1\right) \right\} \\ &\quad \vee \sup_{0 \leq \theta \leq \pi/4} \left\{ \left(1 - \frac{y(\theta)}{x(\theta)}\right) B(z) + g\left(1, \frac{y(\theta)}{x(\theta)}\right) \right\}, \end{aligned}$$

where we use that  $x$  and  $y$  are nonzero, respectively, on the intervals  $[0, \pi/4]$  and  $[\pi/4, \pi/2]$ . Collecting the terms on the right-hand side and using homogeneity, we obtain that

$$\begin{aligned} 0 &= \max \left\{ \sup_{\pi/4 \leq \theta \leq \pi/2} \frac{1}{y(\theta)} \{-x(\theta)A(z) - y(\theta)B(z) + g(x(\theta), y(\theta))\}, \right. \\ &\quad \left. \sup_{0 \leq \theta \leq \pi/4} \frac{1}{x(\theta)} \{-x(\theta)A(z) - y(\theta)B(z) + g(x(\theta), y(\theta))\} \right\}. \end{aligned} \quad (5.3)$$

The expressions inside the supremums in (5.3) are continuous functions of  $\theta$  over closed intervals. Hence, there exists  $\theta_z \in [0, \pi/2]$  such that

$$0 = -x(\theta_z)A(z) - y(\theta_z)B(z) + g(x(\theta_z), y(\theta_z)) = \sup_{0 \leq \theta \leq \pi/2} \{-x(\theta)A(z) - y(\theta)B(z) + g(x(\theta), y(\theta))\},$$

where the second equality is due to  $g \leq g_z$ .  $\square$

**Corollary 5.3.**

$$B(z) = \sup_{0 \leq s < \infty} \{-sA(z) + g(s, 1)\} \quad \text{for } z \in (-\underline{\alpha}, \underline{\beta}). \quad (5.4)$$

*Proof.* Let  $S > 0$ . The set  $\{(s, 1) : 0 \leq s \leq S\} \cup \{(S, t) : 0 \leq t \leq 1\}$  is the image of a curve  $\theta \mapsto (r(\theta) \cos \theta, r(\theta) \sin \theta)$  for  $[0, \pi/2]$  with continuous and positive  $r$ . Hence, by Lemma 5.2,

$$0 = \max\left\{\sup_{0 \leq s \leq S} \{g(s, 1) - g_z(s, 1)\}, \sup_{0 \leq t \leq 1} \{g(S, t) - g_z(S, t)\}\right\}. \quad (5.5)$$

Using homogeneity and Lemma 5.1, we observe that

$$g(S, t) - g_z(S, t) = g(S, t) - SA(z) - tB(z) \leq S(g(1, 1/S) - A(z)) \rightarrow -\infty \quad \text{as } S \rightarrow \infty. \quad (5.6)$$

Hence, the second supremum in (5.5) can be dropped provided that  $S$  is sufficiently large, which results in  $0 = \sup_{0 \leq s \leq S} \{g(s, 1) - g_z(s, 1)\}$ . This equality remains valid if  $S$  is replaced with  $\infty$  by (5.6) with  $t = 1$ . Rearranging terms gives (5.4).  $\square$

*Proof of Theorem 2.1.* Define the function  $\gamma : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$  by  $\gamma(s) = -g(s, 1)$  for  $s \geq 0$  and  $\gamma(s) = \infty$  for  $s < 0$ . By Proposition 3.1,  $\gamma$  is nonincreasing, continuous and convex on  $[0, \infty)$  and completely determines  $g$ . Let  $\gamma^*$  denote the convex conjugate of  $\gamma$ , that is,

$$\gamma^*(x) = \sup_{s \in \mathbb{R}} \{sx - \gamma(s)\} = \sup_{s \geq 0} \{sx - \gamma(s)\} \quad \text{for } x \in \mathbb{R}. \quad (5.7)$$

Let  $f$  be the function whose graph is the image of the curve  $z \mapsto (-A(z), B(z))$ . That is,  $f$  is defined on the interval  $(-A(-\underline{\alpha}), -A(\underline{\beta}))$  and is given by the formula  $f(x) = B \circ A^{-1}(-x)$ . By Corollary 5.3,

$$f(x) = \sup_{0 \leq s < \infty} \{sx - \gamma(s)\} \quad \text{for } x \in (-A(-\underline{\alpha}), -A(\underline{\beta})) \quad (5.8)$$

Comparison of (5.7) and (5.8) shows that  $\gamma^*$  coincides with  $f$  on  $(-A(-\underline{\alpha}), -A(\underline{\beta}))$ . Since  $\gamma$  is a lower semi-continuous, proper convex function on the real line, by the Fenchel-Moreau theorem,  $\gamma$  equals the convex conjugate of  $\gamma^*$ , hence,

$$\gamma(s) = \sup_{x \in \mathbb{R}} \{sx - \gamma^*(x)\} \quad \text{for } s \in \mathbb{R} \quad (5.9)$$

To prove the result, we need to show the supremum in (5.9) can be taken over the interval  $(-A(-\underline{\alpha}), -A(\underline{\beta}))$  instead of the real line. It is clear from (5.7) that  $\gamma^*$  is nondecreasing and is bounded below by  $-\gamma(0) = g(0, 1) = B(-\underline{\alpha})$ . Since  $\gamma^*$  agrees with  $f$  on  $(-A(-\underline{\alpha}), -A(\underline{\beta}))$ ,

$$\begin{aligned} B(-\underline{\alpha}) &\leq \gamma^*(-A(-\underline{\alpha})) \leq \lim_{x \downarrow -A(-\underline{\alpha})} f(x) \\ &= \lim_{x \downarrow -A(-\underline{\alpha})} B \circ A^{-1}(-x) = \lim_{z \rightarrow -\underline{\alpha}} B(z) = B(-\underline{\alpha}), \end{aligned} \quad (5.10)$$

where we used continuity of  $A^{-1}$  and  $B$ . Hence,  $\gamma^*(x) = B(-\underline{\alpha})$  for  $x \leq -A(-\underline{\alpha})$ . On the other hand, if  $x > -A(\underline{\beta}) = -g(1, 0)$  then  $\gamma^*(x) = \infty$  by (5.7) because

$$\lim_{s \rightarrow \infty} sx - \gamma(s) = \lim_{s \rightarrow \infty} s(x + g(1, 1/s)) = \infty.$$

Finally, we compute  $\gamma^*$  at  $-A(\underline{\beta})$ . Being a convex conjugate,  $\gamma^*$  is lower semi-continuous. Since  $\gamma^*$  is also nondecreasing,  $\lim_{y \uparrow x} \gamma^*(y) = \gamma^*(x)$  for any  $x \in \mathbb{R}$ . Then, proceeding as in (5.10),

$$\gamma^*(-A(\underline{\beta})) = \lim_{x \uparrow -A(\underline{\beta})} f(x) = B(\underline{\beta}).$$

We conclude that the function  $x \mapsto sx - \gamma^*(x)$  is increasing for  $x \leq -A(-\underline{\alpha})$  and is  $-\infty$  for  $x > -A(\underline{\beta})$ . Moreover, the left- and right-hand limits agree with the value of the function at  $-A(\underline{\beta})$  and  $-A(-\underline{\alpha})$ , respectively. Hence, by (5.9),

$$\gamma(s) = \sup_{s \in (-A(-\underline{\alpha}), -A(\underline{\beta}))} \{sx - \gamma^*(x)\} = \sup_{z \in (-\underline{\alpha}, \underline{\beta})} \{-sA(z) - B(z)\} = - \inf_{z \in (-\underline{\alpha}, \underline{\beta})} \{sA(z) + B(z)\},$$

which implies (2.1).  $\square$

*Proof of Theorem 2.2.* Introduce  $\delta > 0$  and let  $\varphi : \mathbb{R}_+^{\mathbb{N}} \rightarrow \mathbb{R}_+^{\mathbb{N}}$  denote the map  $(c_n)_{n \in \mathbb{N}} \mapsto (c_n \vee \delta)_{n \in \mathbb{N}}$ . Because  $\varphi$  commutes with the shift  $\tau_1$ ,  $\varphi(\mathbf{a})$  and  $\varphi(\mathbf{b})$  are stationary sequences in  $(0, \infty)$ . Moreover, for each  $k, l \in \mathbb{N}$ , the distribution  $\mu_\delta$  of  $(\varphi(\mathbf{a}), \varphi(\mathbf{b}))$  is ergodic with respect to  $\tau_k \times \tau_l$ . To see this, suppose that  $B = (\tau_k \times \tau_l)^{-1}(B)$  for some  $k, l \in \mathbb{N}$  and Borel set  $B \subset \mathbb{R}_+^{\mathbb{N}} \times \mathbb{R}_+^{\mathbb{N}}$ . Then  $(\varphi \times \varphi)^{-1}(B) = (\varphi \times \varphi)^{-1}((\tau_k \times \tau_l)^{-1}(B)) = (\tau_k \times \tau_l)^{-1}((\varphi \times \varphi)^{-1}(B))$ . Hence, by the ergodicity of  $\mu$ , we get  $\mu_\delta(B) = \mu((\varphi \times \varphi)^{-1}(B)) \in \{0, 1\}$ .

Let  $\alpha_\delta$  and  $\beta_\delta$  denote the marginal distributions of  $\varphi(\mathbf{a})$  and  $\varphi(\mathbf{b})$ , respectively. Then  $\alpha_\delta = \beta_\delta = \delta$ . Applying Theorem 2.1 gives

$$g^{\alpha_\delta, \beta_\delta}(s, t) = \inf_{z \in (-\delta, \delta)} \left\{ s \mathbb{E} \left[ \frac{1}{a \vee \delta + z} \right] + t \mathbb{E} \left[ \frac{1}{b \vee \delta - z} \right] \right\}. \quad (5.11)$$

Since  $\mathbf{P}_{\mathbf{a}, \mathbf{b}}$  stochastically dominates  $\mathbf{P}_{\varphi(\mathbf{a}), \varphi(\mathbf{b})}$ , we have  $g^{\alpha_\delta, \beta_\delta}(s, t) \leq g^{\alpha, \beta}(s, t)$  for  $s, t \geq 0$ . Using this and (5.11), we obtain

$$g^{\alpha, \beta}(s, t) \geq \inf_{z \in (-\delta', \delta')} \left\{ s \mathbb{E} \left[ \frac{1}{a \vee \delta' + z} \right] + t \mathbb{E} \left[ \frac{1}{b \vee \delta' - z} \right] \right\},$$

where we fix  $\delta' > \delta$ . Because the expression inside the infimum is continuous in  $z$ , letting  $\delta \downarrow 0$  yields  $g^{\alpha, \beta}(s, t) \geq s \mathbb{E}[(a \vee \delta')^{-1}] + t \mathbb{E}[(b \vee \delta')^{-1}]$  for  $s, t \geq 0$ . Then, by monotone convergence, letting  $\delta' \rightarrow 0$  results in

$$g^{\alpha, \beta}(s, t) \geq s \mathbb{E} \left[ \frac{1}{a} \right] + t \mathbb{E} \left[ \frac{1}{b} \right] \quad \text{for } s, t \geq 0.$$

The opposite inequality is noted after Lemma 4.3.  $\square$

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